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Abstract
We study quantum games with correlated noise through a generalized
quantization scheme. We investigate the effects of memory on quantum games,
such as Prisoner's Dilemma, Battle of the Sexes and Chicken, through three
prototype quantum-correlated channels. It is shown that the quantum player
enjoys an advantage over the classical player for all nine cases considered in
this paper for the maximally entangled case. However, the quantum player can
also outperform the classical player for subsequent cases that can be noted in
the case of the Battle of the Sexes game. It can be seen that the Nash equilibria
do not change for all the three games under the effect of memory.

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1. Introduction

The study of quantum games combines the laws of quantum mechanics with game theory. It
is interesting to study the games at microscopic level where the laws of quantum mechanics
dictates the dynamics. Quantum games offer additional strategies to the players and resolve
dilemmas that occur in classical games [1–6]. Quantum theory has already been applied to
a wide variety of games [7–11] and shown to be experimentally feasible [12]. Additionally,
quantum games offer a new paradigm for exploring the fascinating world of quantum
information [13–15]. Meyer [16] has also pointed out the connection between quantum
games and quantum information processing.

   In the earlier work on quantum games, for simplicity, the role of channels is mostly
ignored. In a realistic setup, however, the flow of information between players and arbiter
is subject to interaction with the environment. Quantum entanglement, which is one of
the interesting features of quantum mechanics, plays a crucial role in quantum information
processing. When quantum information processing is performed in the real world, decoherence
caused by an external environment is inevitable. In other words, the influence of an external
environmental system on the entanglement cannot be ignored. Recently, decoherence effects
in quantum games have been studied [17].

Later, interest has been developed to extend the theory of quantum channels to encompass memory effects [18, 19]. There are timescales for which successive uses of channel are correlated and memory effects need to be taken into account. Quantum computing in the presence of noise is possible with the use of decoherence-free subspaces [20] and quantum error correction [21]. Studies concerning quantum games in the presence of decoherence and correlated noise have produced interesting results. Chen *et al* [17] have shown that in the case of the game Prisoner's Dilemma, the Nash equilibria are not changed by the effects of decoherence for maximally entangled states incorporating three prototype decoherence channels. Recently, Nawaz and Toor [22] have shown for the quantum games based on quantum-correlated phase-damping channel that the quantum player only enjoys an advantage over the classical player when both the initial quantum state and the measurement basis are in entangled form. It is also shown that for maximum correlation the effects of decoherence diminish and it behaves as a noiseless game. Recently, Cao *et al* [23] have investigated the effect of quantum noise on a multiplayer quantum game. They have shown that in a maximally entangled case a special Nash equilibrium appears for a specific range of quantum noise parameter.

In this paper, we study the quantum games based on three prototype quantum-correlated channels (QCC) parameterized by a memory factor μ which measures the degree of correlations, in the context of generalized quantization scheme for nonzero sum games [24]. We identify four different regimes on the basis of initial state and measurement basis entanglement parameters, $\gamma \in [0, \pi/2]$ and $\delta \in [0, \pi/2]$, respectively. For these four regimes, we study the role of decoherence parameter $p \in [0, 1]$ and memory parameter $\mu \in [0, 1]$ for three quantum games. Here, $\delta = 0$ means that the measurement basis are unentangled and $\delta = \pi/2$ means that it is maximally entangled, $\gamma = 0$ means that the game is initially unentangled and $\gamma = \pi/2$ means that it is maximally entangled. Whereas the lower and upper limits of p correspond to a fully coherent and fully decohered system, respectively. Furthermore, the lower and upper limits of μ correspond to a memoryless and maximum memory (degree of correlation) cases, respectively. It is shown that for $\gamma = \delta = 0$, with the decoherence and noise parameters $p_1 = p_2 = 0$ and $\mu_1 = \mu_2 = 0$, respectively, the game reduces to the classical one for all the cases discussed in this paper. In Prisoner's Dilemma game, when $\gamma \neq 0, \delta = 0$, it is interesting to note that though the initial state is entangled, the quantum player has no advantage over the classical player in Prisoner's Dilemma and Chicken games. The same happens for the case of $\gamma = 0, \delta \neq 0$. Interesting aspects of these cases arise for game's entanglement parameter, γ, and measurement basis entanglement parameter, δ. In the case of the Battle of the Sexes game, for $\delta = 0, \gamma \neq 0$ and $\gamma = 0, \delta \neq 0$, it is seen that the quantum player is better off for $p > 0$ for amplitude-damping and depolarizing channels, respectively. For $\gamma = \delta = \pi/2$, the quantum player remains better off for all values of the decoherence parameter p and the memory parameter μ against a player restricted to classical strategies, for all the nine cases considered.

2. Quantum channels with memory

Several investigations concern the transmission of quantum information from one party (Alice) to another (Bob) through a communication channel. In the most basic configuration the information is encoded in qubits. If the qubits are perfectly protected from environmental influence, Bob receives them in the same state as prepared by Alice. In a more realistic case, however, the qubits have a nontrivial dynamics during the transmission because of their interaction with the environment [25]. Therefore, Bob receives a set of distorted qubits because of the disturbing action of the channel. Recently, the study of quantum channels with memory
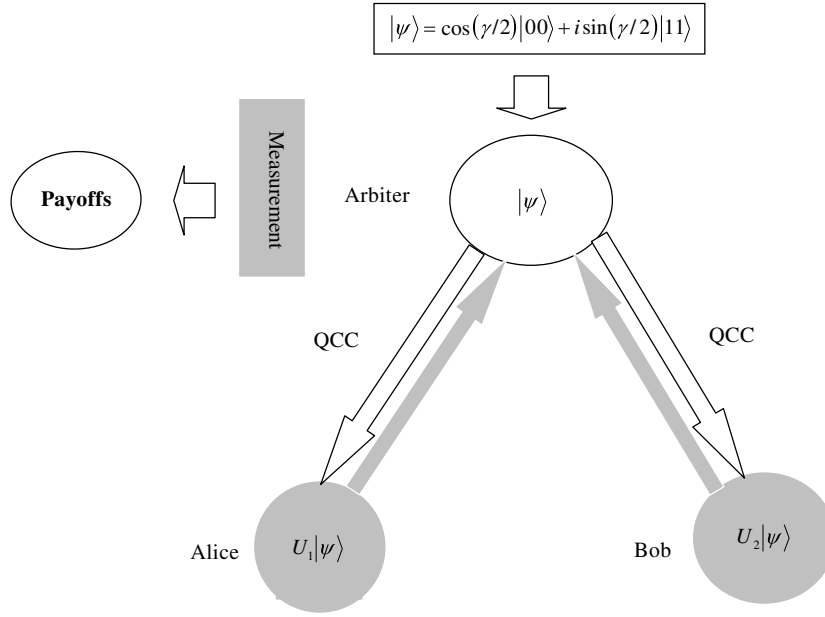
Figure 1. Schematic diagram of the model.

has attracted a lot of attention [18, 19, 26] in the hope that by entangling multiple uses of the channel, a larger amount of classical information per use could be reliably transmitted. Early works in this direction were devoted, mainly, to memoryless channels for which consecutive signal transmissions through the channel are not correlated. Correlated noise, also referred as memory in the literature, acts on consecutive uses of the channels. However, in general, one may want to encode classical data into entangled strings or consecutive uses of the channel may be correlated to each other. Hence, we are dealing with a strongly correlated quantum system, the correlation of which results from the memory of the channel itself. In our model, Alice and Bob, each uses individual channels to communicate with the arbiter of the game. Alice's channel is correlated in time and therefore has a memory. The two uses of the channel, i.e. the first passage (from the arbiter) and the second passage (back to the arbiter) are correlated. A similar situation occurs for Bob as depicted in figure 1. We consider here different noise models based on phase-damping, amplitude-damping and depolarizing channels.

The action of transmission channels is described by Kraus operators which satisfy $\sum_{i=0}^{1} A_i^\dagger A_i = 1$. In the operator sum representation, the dephasing process can be expressed as [25]

$$\rho_f = \sum_{i=0}^{1} A_i \rho_{\text{in}} A_i^\dagger, \tag{1}$$

where ρ_{in} represents the initial density matrix for quantum state and

$$A_0 = \sqrt{1 - \frac{p}{2}} I, \qquad A_1 = \sqrt{\frac{p}{2}} \sigma_z \tag{2}$$

are the Kraus operators, I is the identity operator, p is the decoherence parameter and σ_z is the Pauli matrix. Let N qubits are allowed to pass through such a channel, then equation (1)

becomes [27]

$$\rho_f = \sum_{k_1,\dots,k_n=0}^{1} \left(A_{k_n} \otimes \cdots A_{k_1}\right) \rho_{\text{in}} \left(A_{k_1}^{\dagger} \otimes \cdots A_{k_n}^{\dagger}\right). \tag{3}$$

Now if the noise is correlated with memory of degree μ, then the action of the channel on two consecutive qubits is given by the Kraus operator [18]

$$A_{ij} = \sqrt{p_i[(1-\mu)p_j + \mu\delta_{ij}]}\sigma_i \otimes \sigma_j, \tag{4}$$

where σ_i and σ_j are the usual Pauli matrices with indices i and j running from 0 to 3 and μ is the memory parameter. The above expression means that with probability $1 - \mu$ the noise is uncorrelated, whereas with probability μ the noise is correlated as illustrated in the equations below. Physically, the parameter μ is determined by the relaxation time of the channel when a qubit passes through it. In order to remove correlations, one can wait until the channel has relaxed to its original state before sending the next qubit, however this lowers the rate of information transfer. Thus, it is necessary to consider the performance of the channel for arbitrary values of μ to reach a compromise between various factors which determine the final rate of information transfer. Thus, during passing through the channel any two consecutive qubits undergo random independent (uncorrelated) errors with probability $1 - \mu$ and identical (correlated) errors with probability μ. This should be the case if the channel has a memory depending on its relaxation time and if we stream the qubits through it. A quantum dephasing channel (Pauli Z channel) with uncorrelated noise (memoryless channel) can be defined as the one specified by the Kraus operators

$$Z_{ij}^{u} = \sqrt{p_i p_j}\sigma_i \otimes \sigma_j, \qquad i, j = 0, 3, \tag{5}$$

and the one with correlated noise (channel with memory) by

$$Z_{kk}^{c} = \sqrt{p_k}\sigma_k \otimes \sigma_k, \qquad k = 0, 3. \tag{6}$$

The action of a depolarizing channel with memory can be expressed as

$$\pi \to \rho = \Phi(\pi) = (1-\mu)\sum_{i,j=0}^{3} D_{ij}^{u}\pi D_{ij}^{u\dagger} + \mu\sum_{k=0}^{3} D_{kk}^{c}\pi D_{kk}^{c\dagger}, \tag{7}$$

where $0 \leqslant \mu \leqslant 1$. With probability $1 - \mu$ the noise is uncorrelated and completely specified by the Kraus operators

$$D_{ij}^{u} = \sqrt{p_i p_j}\sigma_i \otimes \sigma_j, \tag{8}$$

and with probability μ the noise is correlated and specified by

$$D_{kk}^{c} = \sqrt{p_k}\sigma_i \otimes \sigma_j, \tag{9}$$

where $0 \leqslant p \leqslant 1$, $p_0 = (1 - p)$, $p_1 = p_2 = p_3 = p/3$ and $\sigma_0, \sigma_1, \sigma_2$ and σ_3 are the identity and Pauli matrices, respectively. However, we note that a quantum amplitude-damping channel with uncorrelated noise can be defined as the one specified by the following Kraus operators:

$$A_{00}^{u} = A_0 \otimes A_0, \qquad A_{01}^{u} = A_0 \otimes A_1, \qquad A_{10}^{u} = A_1 \otimes A_0, \qquad A_{11}^{u} = A_1 \otimes A_1, \tag{10}$$

$$A_0 = \begin{bmatrix} \cos\chi & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ \sin\chi & 0 \end{bmatrix}. \tag{11}$$

However, the Kraus operators for a quantum amplitude-damping channel with correlated noise are given by Yeo and Skeen [19] as

$$A_{00}^{c} = \begin{bmatrix} \cos\chi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A_{11}^{c} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin\chi & 0 & 0 & 0 \end{bmatrix}, \tag{12}$$

where $0 \leqslant \chi \leqslant \pi/2$ and is related to the decoherence parameter as

$$\cos^2 \chi = 1 - p, \qquad \sin^2 \chi = p. \tag{13}$$

It is clear that A_{00}^c cannot be written as a tensor product of two 2×2 matrices. This gives rise to the typical spooky action of the channel: $|01\rangle$ and $|10\rangle$, and any linear combination of them, and $|11\rangle$ will go through the channel undisturbed, but not $|00\rangle$. The action of this non-unital channel is given by

$$\pi \rightarrow \rho = \Phi(\pi) = (1 - \mu) \sum_{i,j=0}^{1} A_{ij}^u \pi A_{ij}^{u\dagger} + \mu \sum_{k=0}^{1} A_{kk}^c \pi A_{kk}^{c\dagger}. \tag{14}$$

The protocol for quantum games in the presence of correlated noise is developed by Nawaz and Toor [22]. We consider that an initial entangled state is prepared by the arbiter and passed on to the players through three prototype quantum-correlated channels (as shown in figure 1), i.e. Alice and Bob each uses individual channels to communicate with the arbiter of the game. Alice's channel is correlated in time (and therefore has a memory), i.e. the two uses of the channel are correlated. On receiving the quantum state from the arbiter, the players apply their local operators (strategies) and return it back to the arbiter through QCC. The arbiter then performs the measurement and announces their payoffs. Let the game start with the initial quantum state given as

$$|\psi_{\text{in}}\rangle = \cos \frac{\gamma}{2} |00\rangle + i \sin \frac{\gamma}{2} |11\rangle, \tag{15}$$

where $0 \leqslant \gamma \leqslant \pi/2$ corresponds to the entanglement of the initial state. The strategies of the players in the generalized quantization scheme are represented by the unitary operator U_i of the form [24]

$$U_i = \cos \frac{\theta_i}{2} R_i + \sin \frac{\theta_i}{2} P_i, \tag{16}$$

where $i = 1$ or 2 and R_i, P_i are the unitary operators defined as

$$\begin{aligned} R_i|0\rangle &= e^{i\alpha_i}|0\rangle, & R_i|1\rangle &= e^{-i\alpha_i}|1\rangle, \\ P_i|0\rangle &= e^{i(\frac{\pi}{2}-\beta_i)}|1\rangle, & P_i|1\rangle &= e^{i(\frac{\pi}{2}+\beta_i)}|0\rangle, \end{aligned} \tag{17}$$

where $0 \leqslant \theta \leqslant \pi$ and $-\pi \leqslant \alpha, \beta \leqslant \pi$. Under the generalized quantization scheme with three parameter strategies, the initial state given in equation (15) transforms to

$$\rho_f = (U_1 \otimes U_2)\rho_{\text{in}}(U_1 \otimes U_2)^{\dagger}, \tag{18}$$

where $\rho_{\text{in}} = |\psi_{\text{in}}\rangle \langle \psi_{\text{in}}|$ is the density matrix for the quantum state. The operators used by the arbiter to determine the payoff for Alice and Bob are

$$P = \$_{00} P_{00} + \$_{01} P_{01} + \$_{10} P_{10} + \$_{11} P_{11}, \tag{19}$$

where

$$\begin{aligned} P_{00} &= |\psi_{00}\rangle\langle\psi_{00}|, & |\psi_{00}\rangle &= \cos \frac{\delta}{2}|00\rangle + i \sin \frac{\delta}{2}|11\rangle, \\[6pt] P_{11} &= |\psi_{11}\rangle\langle\psi_{11}|, & |\psi_{11}\rangle &= \cos \frac{\delta}{2}|11\rangle + i \sin \frac{\delta}{2}|00\rangle, \\[6pt] P_{10} &= |\psi_{10}\rangle\langle\psi_{10}|, & |\psi_{10}\rangle &= \cos \frac{\delta}{2}|10\rangle - i \sin \frac{\delta}{2}|01\rangle, \\[6pt] P_{01} &= |\psi_{01}\rangle\langle\psi_{01}|, & |\psi_{01}\rangle &= \cos \frac{\delta}{2}|01\rangle - i \sin \frac{\delta}{2}|10\rangle, \end{aligned} \tag{20}$$

with $0 \leqslant \delta \leqslant \pi/2$ and $\$_{ij}$ are the elements of the payoff matrix in the ith row and jth column of classical games as given in appendix A. In the generalized quantization scheme for the three set of parameters, the players' payoffs read

$$\$^A(\theta_i, \alpha_i, \beta_i) = \text{Tr}(P_A \rho_f), \qquad \$^B(\theta_i, \alpha_i, \beta_i) = \text{Tr}(P_B \rho_f), \tag{21}$$

where Tr represents the trace of the matrix. Using equations (4)–(9), (14), (19) and (21), the payoffs of the two players, when both channels (first and second) are amplitude-damping, are given by

$$
\begin{aligned}
\$(\theta_i, \alpha_i, \beta_i) = {} & c_1 c_2 \big[\eta_1^A \$_{00} + \chi_1^A \$_{11} + \Delta_1^A (\$_{01} + \$_{10}) + (\$_{11} - \$_{00}) \chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(10)} \xi \cos 2(\alpha_1 + \alpha_2) \big] \\
& + s_1 s_2 \big[\eta_2^A \$_{00} + \chi_2^A \$_{11} + \Delta_2^A (\$_{01} + \$_{10}) - (\$_{11} - \$_{00}) \chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(10)} \xi \cos 2(\beta_1 + \beta_2) \big] \\
& + s_1 c_2 \big[\eta_3^A \$_{00} + \chi_3^A \$_{11} + \Delta_3^A \$_{01} + \Delta_4^A \$_{10} + (\$_{01} - \$_{10}) \chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(b)} \xi \cos 2(\alpha_2 - \beta_1) \big] \\
& + c_1 s_2 \big[\eta_3^A \$_{00} + \chi_3^A \$_{11} + \Delta_4^A \$_{01} + \Delta_3^A \$_{10} - (\$_{01} - \$_{10}) \chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(b)} \xi \cos 2(\alpha_1 - \beta_2) \big] \\
& - \tfrac{1}{4} \big[\sin(\delta) \sin(\theta_1) \sin(\theta_2) \chi_{\mu_2}^{(10)} (\$_{00} - \$_{11}) \Delta_5^A \sin(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \big] \\
& - \tfrac{1}{4} [\sin(\gamma) \sin(\theta_1) \sin(\theta_2) (\eta_4 \$_{00} + \chi_4 \$_{11}) \Delta_5^A \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)] \\
& - \tfrac{1}{4} [\sin(\gamma) \sin(\theta_1) \sin(\theta_2) (\$_{01} + \$_{10}) \Delta_6^A \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)] \\
& + \tfrac{1}{4} \big[\sin(\delta) \sin(\theta_1) \sin(\theta_2) \chi_{\mu_2}^{(b)} (\$_{01} - \$_{10}) \Delta_5^A \sin(\alpha_1 - \alpha_2 + \beta_1 - \beta_2) \big]. \tag{22}
\end{aligned}
$$

The payoffs of the two players, when both channels are depolarizing, are given as

$$
\begin{aligned}
\$(\theta_i, \alpha_i, \beta_i) = {} & c_1 c_2 \big[\eta^D \$_{00} + \chi^D \$_{11} + \Delta^D (\$_{01} + \$_{10}) + (\$_{00} - \$_{11}) \big(\Delta_{\mu 2}^4 - \tfrac{2}{3} \mu_2 p_2 \big) \\
& \times \Delta_{\mu 1}^3 \xi \cos 2(\alpha_1 + \alpha_2) \big] + s_1 s_2 \big[\chi^D \$_{00} + \eta^D \$_{11} + \Delta^D (\$_{01} + \$_{10}) \\
& - (\$_{00} - \$_{11}) \big(\Delta_{\mu 2}^4 + \tfrac{2}{3} \mu_2 p_2 \big) \Delta_{\mu 1}^3 \xi \cos 2(\beta_1 + \beta_2) \big] + s_1 c_2 \big[\eta^D \$_{10} + \chi^D \$_{01} \\
& + \Delta^D (\$_{00} + \$_{11}) + (\$_{01} - \$_{10}) \big(\Delta_{\mu 2}^4 - \tfrac{2}{3} \mu_2 p_2 \big) \Delta_{\mu 1}^3 \xi \cos 2(\alpha_2 - \beta_1) \big] \\
& + c_1 s_2 \big[\eta^D \$_{01} + \chi^D \$_{10} + \Delta^D (\$_{00} + \$_{11}) - (\$_{01} - \$_{10}) \big(\Delta_{\mu 2}^4 + \tfrac{2}{3} \mu_2 p_2 \big) \\
& \times \Delta_{\mu 1}^3 \xi \cos 2(\alpha_1 - \beta_2) \big] - \big(\tfrac{1}{4} \Delta_{\mu 1}^3 \Delta_{\mu 2}^1 + \tfrac{1}{2} \Delta_{\mu 1}^3 \Delta_{\mu 2}^2 - \tfrac{1}{4} \Delta_{\mu 1}^3 \Delta_{\mu 2}^3 \big)(\$_{00} + \$_{11}) \\
& \times \sin(\gamma) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) - \big(\tfrac{1}{4} \Delta_{\mu 2}^4 - \tfrac{1}{6} \mu_2 p_2 \big) \\
& \times (\$_{00} - \$_{11}) \eta_{1DP} \sin(\delta) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \\
& + \big(\tfrac{1}{4} \Delta_{\mu 1}^3 \Delta_{\mu 2}^1 - \tfrac{1}{2} \Delta_{\mu 1}^3 \Delta_{\mu 2}^2 + \tfrac{1}{4} \Delta_{\mu 1}^3 \Delta_{\mu 2}^3 \big)(\$_{01} + \$_{10}) \sin(\gamma) \sin(\theta_1) \sin(\theta_2) \\
& \times \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) - \big(\tfrac{1}{4} \Delta_{\mu 2}^4 - \tfrac{1}{6} \mu_2 p_2 \big)(\$_{01} - \$_{10}) \eta_{1DP} \sin(\delta) \sin(\theta_1) \\
& \times \sin(\theta_2) \sin(\alpha_1 - \alpha_2 + \beta_1 - \beta_2) \big]. \tag{23}
\end{aligned}
$$

The payoffs of the two players, when both channels are phase-damping, are given by

$$
\begin{aligned}
\$(\theta_i, \alpha_i, \beta_i) = {} & c_1 c_2 \big[\eta^P \$_{00} + \chi^P \$_{11} + (\$_{00} - \$_{11}) \mu_p^{(1)} \mu_p^{(2)} \xi \cos 2(\alpha_1 + \alpha_2) \big] \\
& + s_1 s_2 \big[\eta^P \$_{11} + \chi^P \$_{00} - (\$_{00} - \$_{11}) \mu_p^{(1)} \mu_p^{(2)} \xi \cos 2(\beta_1 + \beta_2) \big] \\
& + s_1 c_2 \big[\eta^P \$_{10} + \chi^P \$_{01} + (\$_{10} - \$_{01}) \mu_p^{(1)} \mu_p^{(2)} \xi \cos 2(\alpha_2 - \beta_1) \big] \\
& + c_1 s_2 \big[\eta^P \$_{01} + \chi^P \$_{10} - (\$_{10} - \$_{01}) \mu_p^{(1)} \mu_p^{(2)} \xi \cos 2(\alpha_1 - \beta_2) \big] \\
& + \frac{\mu_p^{(2)}}{4} (\$_{00} - \$_{11}) \sin(\delta) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \\
& + \frac{\mu_p^{(1)}}{4} (-\$_{00} - \$_{11} + \$_{01} + \$_{10}) \sin(\gamma) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) \\
& + \frac{\mu_p^{(2)}}{4} (\$_{10} - \$_{01}) \sin(\delta) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 - \alpha_2 + \beta_1 - \beta_2). \tag{24}
\end{aligned}
$$

The payoffs of the two players, when first channel is phase-damping and second channel is amplitude-damping, are given by

$$\$(\theta_i, \alpha_i, \beta_i) = c_1 c_2 \big[\eta_1^{PA}\$_{00} + \chi_1^{PA}\$_{11} + \Delta_1^{PA}(\$_{01}+\$_{10}) + (\$_{00} - \$_{11})\mu_p^{(1)}\chi_{\mu2}^{(10)}\xi \cos 2(\alpha_1+\alpha_2)\big]$$

$$+ s_1 s_2\big[\eta_2^{PA}\$_{00} + \chi_2^{PA}\$_{11} + \Delta_2^{PA}(\$_{01}+\$_{10}) - (\$_{00}-\$_{11})\mu_p^{(1)}\chi_{\mu2}^{(10)}\xi$$

$$\times \cos 2(\beta_1+\beta_2)\big] + s_1 c_2\big[\Delta_3^{PA}\$_{01} + \Delta_4^{PA}\$_{10} + \eta_3^{PA}\$_{00} + \chi_3^{PA}\$_{11}$$

$$+ (\$_{01}-\$_{10})\mu_p^{(1)}\chi_{\mu2}^{(b)}\xi\cos 2(\alpha_2-\beta_1)\big] + c_1 s_2\big[\Delta_4^{PA}\$_{01} + \Delta_3^{PA}\$_{10} + \eta_3^{PA}\$_{00}$$

$$+ \chi_3^{PA}\$_{11} - (\$_{01}-\$_{10})\mu_p^{(1)}\chi_{\mu2}^{(b)}\xi\cos 2(\alpha_1-\beta_2)\big] - \tfrac{1}{4}\big(\chi_{\mu2}^{(00)} + \chi_{\mu2}^{(11)} - 2\chi_{\mu2}^{(a)}\big)$$

$$\times (\$_{00}-\$_{11})\mu_p^{(1)}\sin(\gamma)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1+\alpha_2-\beta_1-\beta_2)$$

$$+ \tfrac{1}{4}\chi_{\mu2}^{(10)}(\$_{00}-\$_{11})\sin(\delta)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1+\alpha_2+\beta_1+\beta_2)$$

$$+ \tfrac{1}{4}\mu_p^{(1)}(\chi_{\mu2}^{(b)} + \chi_{\mu2}^{(01)})(\$_{01}+\$_{10})\sin(\gamma)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1+\alpha_2-\beta_1-\beta_2)$$

$$+ \tfrac{1}{4}\chi_{\mu2}^{(b)}(\$_{01}-\$_{10})\sin(\delta)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1-\alpha_2+\beta_1-\beta_2). \qquad (25)$$

The payoffs of the two players, when first channel is amplitude-damping and second channel is phase-damping, are given by

$$\$(\theta_i, \alpha_i, \beta_i) = c_1 c_2\big[\eta^{AP}\$_{00} + \chi^{AP}\$_{11} + \Delta^{AP}(\$_{01}+\$_{10}) + (\$_{00}-\$_{11})\mu_p^{(2)}\chi_{\mu1}^{(10)}\xi\cos 2(\alpha_1+\alpha_2)\big]$$

$$+ s_1 s_2\big[\chi^{AP}\$_{00} + \eta^{AP}\$_{11} + \Delta^{AP}(\$_{01}+\$_{10}) - (\$_{00}-\$_{11})\mu_p^{(2)}\chi_{\mu1}^{(10)}\xi$$

$$\times \cos 2(\beta_1+\beta_2)\big] + s_1 c_2\big[\eta^{AP}\$_{10} + \chi^{AP}\$_{01} + \Delta^{AP}(\$_{00}+\$_{11})$$

$$+ (\$_{01}-\$_{10})\mu_p^{(2)}\chi_{\mu1}^{(10)}\xi\cos 2(\alpha_2-\beta_1)\big] + c_1 s_2\big[\eta^{AP}\$_{01} + \chi^{AP}\$_{10}$$

$$+ \Delta^{AP}(\$_{00}+\$_{11}) - (\$_{01}-\$_{10})\mu_p^{(2)}\chi_{\mu1}^{(10)}\xi\cos 2(\alpha_1-\beta_2)\big]$$

$$- \tfrac{1}{4}\chi_{\mu1}^{(10)}(\$_{00}+\$_{11})\sin(\gamma)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1+\alpha_2-\beta_1-\beta_2)$$

$$+ \tfrac{1}{4}\mu_p^{(2)}\eta_{1AD}(\$_{00}+\$_{11})\sin(\delta)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1+\alpha_2+\beta_1+\beta_2)$$

$$+ \tfrac{1}{4}\chi_{\mu1}^{(10)}(\$_{01}+\$_{10})\sin(\gamma)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1+\alpha_2-\beta_1-\beta_2)$$

$$+ \tfrac{1}{4}\mu_p^{(2)}\eta_{1AD}(\$_{01}-\$_{10})\sin(\delta)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1-\alpha_2+\beta_1-\beta_2). \qquad (26)$$

The payoffs of the two players, when first channel is amplitude-damping and second channel is depolarizing, are given by

$$\$(\theta_i, \alpha_i, \beta_i) = c_1 c_2\big[\eta^{AD}\$_{00} + \chi^{AD}\$_{11} + \Delta^{AD}(\$_{01}+\$_{10}) + (\$_{00}-\$_{11})\big(\Delta_{\mu2}^4 - \tfrac{2}{3}\mu_2 p_2\big)\chi_{\mu1}^{(10)}\xi$$

$$\times \cos 2(\alpha_1+\alpha_2)\big] + s_1 s_2\big[\chi^{AD}\$_{00} + \eta^{AD}\$_{11} + \Delta^{AD}(\$_{01}+\$_{10}) - (\$_{00}-\$_{11})$$

$$\times \big(\Delta_{\mu2}^4 - \tfrac{2}{3}\mu_2 p_2\big)\chi_{\mu1}^{(10)}\xi\cos 2(\alpha_1+\alpha_2)\big] + s_1 c_2\big[\chi^{AD}\$_{01} + \eta^{AD}\$_{10}$$

$$+ \Delta^{AD}(\$_{00}+\$_{11}) + (\$_{01}-\$_{10})\big(\Delta_{\mu2}^4 - \tfrac{2}{3}\mu_2 p_2\big)\chi_{\mu1}^{(10)}\xi\cos 2(\alpha_2-\beta_1)\big]$$

$$+ c_1 s_2\big[\eta^{AD}\$_{01} + \chi^{AD}\$_{10} + \Delta^{AD}(\$_{00}+\$_{11}) - (\$_{01}-\$_{10})\big(\Delta_{\mu2}^4 - \tfrac{2}{3}\mu_2 p_2\big)$$

$$\times \chi_{\mu1}^{(10)}\xi\cos 2(\alpha_1-\beta_2)\big] - \big(\tfrac{1}{4}\chi_{\mu1}^{(10)}\Delta_{\mu2}^1 - \tfrac{1}{2}\chi_{\mu1}^{(10)}\Delta_{\mu2}^2 + \tfrac{1}{4}\chi_{\mu1}^{(10)}\Delta_{\mu2}^3\big)$$

$$\times (\$_{00}+\$_{11})\sin(\gamma)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1+\alpha_2-\beta_1-\beta_2) + \big(\tfrac{1}{4}\Delta_{\mu2}^4 - \tfrac{1}{6}\mu_2 p_2\big)$$

$$\times (\$_{00}-\$_{11})\eta_{1AD}\sin(\delta)\sin(\theta_1)\sin(\theta_2)\sin(\alpha_1+\alpha_2+\beta_1+\beta_2)$$

$$+ \big(\tfrac{1}{4}\chi_{\mu1}^{(10)}\Delta_{\mu2}^1 - \tfrac{1}{2}\chi_{\mu1}^{(10)}\Delta_{\mu2}^2 + \tfrac{1}{4}\chi_{\mu1}^{(10)}\Delta_{\mu2}^3\big)(\$_{01}+\$_{10})\sin(\gamma)\sin(\theta_1)\sin(\theta_2)$$

$$\times \sin(\alpha_1+\alpha_2-\beta_1-\beta_2) + \big(\tfrac{1}{4}\Delta_{\mu2}^4 - \tfrac{1}{6}\mu_2 p_2\big)(\$_{01}-\$_{10})\eta_{1AD}\sin(\delta)$$

$$\times \sin(\theta_1)\sin(\theta_2)\sin(\alpha_1-\alpha_2+\beta_1-\beta_2)\big]. \qquad (27)$$

The payoffs of the two players, when first channel is depolarizing and second channel is amplitude-damping, are given by

$$
\begin{aligned}
\$(\theta_i, \alpha_i, \beta_i) &= c_1 c_2 \Big[\eta_1^{DA} \$_{00} + \chi_1^{DA} \$_{11} + \Delta_1^{DA} (\$_{01} + \$_{10}) + (\$_{00} - \$_{11}) \Delta_{\mu 1}^3 \chi_{\mu 2}^{(10)} \chi_{\mu 1}^{(10)} \xi \\
&\quad \times \cos 2(\alpha_1 + \alpha_2) \Big] + s_1 s_2 \Big[\eta_2^{DA} \$_{00} + \chi_2^{DA} \$_{11} + \Delta_1^{DA} (\$_{01} + \$_{10}) - (\$_{00} - \$_{11}) \\
&\quad \times \Delta_{\mu 1}^3 \chi_{\mu 2}^{(10)} \chi_{\mu 1}^{(10)} \xi \cos 2(\beta_1 + \beta_2) \Big] + s_1 c_2 \Big[\Delta_2^{DA} \$_{01} + \Delta_3^{DA} \$_{10} + \eta_3^{DA} \$_{00} \\
&\quad + \chi_3^{DA} \$_{11} + (\$_{01} - \$_{10}) \Delta_{\mu 1}^3 \chi_{\mu 2}^{(b)} \xi \cos 2(\alpha_2 - \beta_1) + c_1 s_2 \Big[\Delta_3^{DA} \$_{01} + \Delta_2^{DA} \$_{10} \\
&\quad + \eta_3^{DA} \$_{00} + \chi_3^{DA} \$_{11} - (\$_{01} - \$_{10}) \Delta_{\mu 1}^3 \chi_{\mu 2}^{(b)} \xi \cos 2(\alpha_1 - \beta_2) \Big] \\
&\quad - \left(\tfrac{1}{4} \chi_{\mu 2}^{(00)} \Delta_{\mu 1}^3 + \tfrac{1}{4} \Delta_{\mu 1}^3 + \tfrac{1}{4} \Delta_{\mu 1}^3 \chi_{\mu 2}^{(11)} \right) (\$_{00} + \$_{11}) \sin(\gamma) \sin(\theta_1) \sin(\theta_2) \\
&\quad \times \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) - \left(\tfrac{1}{4} \eta_{1DP} \chi_{\mu 2}^{(10)} \right) (\$_{00} - \$_{11}) \sin(\delta) \sin(\theta_1) \sin(\theta_2) \\
&\quad \times \sin(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) - \left(\tfrac{1}{2} \chi_{\mu 2}^{(01)} \Delta_{\mu 1}^3 - \tfrac{1}{4} \chi_{\mu 2}^{(b)} \Delta_{\mu 1}^3 \right) (\$_{01} + \$_{10}) \sin(\gamma) \sin(\theta_1) \\
&\quad \times \sin(\theta_2) \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) - \left(\tfrac{1}{4} \eta_{1DP} \chi_{\mu 2}^{(b)} \right) (\$_{01} - \$_{10}) \sin(\delta) \sin(\theta_1) \\
&\quad \times \sin(\theta_2) \sin(\alpha_1 - \alpha_2 + \beta_1 - \beta_2) \Big].
\end{aligned}
\tag{28}
$$

The payoffs of the two players, when first channel is depolarizing and second channel is phase-damping, are given by

$$
\begin{aligned}
\$(\theta_i, \alpha_i, \beta_i) &= c_1 c_2 \Big[\eta^{DP} \$_{00} + \chi^{DP} \$_{11} + \Delta_{\mu 1}^4 (\$_{01} + \$_{10}) + (\$_{00} - \$_{11}) \Delta_{\mu 1}^3 \mu_p^2 \xi \cos 2(\alpha_1 + \alpha_2) \Big] \\
&\quad + s_1 s_2 \Big[\chi^{DP} \$_{00} + \eta^{DP} \$_{11} + \Delta_{\mu 1}^4 (\$_{01} + \$_{10}) - (\$_{00} - \$_{11}) \Delta_{\mu 1}^3 \mu_p^2 \xi \cos 2(\beta_1 + \beta_2) \Big] \\
&\quad + s_1 c_2 \Big[\chi^{DP} \$_{01} + \eta^{DP} \$_{10} + \Delta_{\mu 1}^4 (\$_{00} + \$_{11}) + (\$_{01} - \$_{10}) \Delta_{\mu 1}^3 \mu_p^2 \xi \cos 2(\alpha_2 - \beta_1) \Big] \\
&\quad + c_1 s_2 \Big[\chi^{DP} \$_{10} + \eta^{DP} \$_{01} + \Delta_{\mu 1}^4 (\$_{00} + \$_{11}) - (\$_{01} - \$_{10}) \Delta_{\mu 1}^3 \mu_p^2 \xi \cos 2(\alpha_1 - \beta_2) \Big] \\
&\quad - \left(\tfrac{1}{4} \Delta_{\mu 1}^3 \right) (\$_{00} + \$_{11}) \sin(\gamma) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) \\
&\quad - \left(\tfrac{1}{4} \eta_{1DP} \mu_p^2 \right) (\$_{00} - \$_{11}) \sin(\delta) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \\
&\quad + \left(\tfrac{1}{4} \Delta_{\mu 1}^3 \right) (\$_{01} + \$_{10}) \sin(\gamma) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) \\
&\quad + \left(\tfrac{1}{4} \eta_{1DP} \mu_p^2 \right) (\$_{01} - \$_{10}) \sin(\delta) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 - \alpha_2 + \beta_1 - \beta_2) \Big].
\end{aligned}
\tag{29}
$$

The payoffs of the two players, when first channel is phase-damping and second channel is depolarizing, are given by

$$
\begin{aligned}
\$(\theta_i, \alpha_i, \beta_i) &= c_1 c_2 \Big[\eta^{PD} \$_{00} + \chi^{PD} \$_{11} + \Delta_{\mu 2}^2 (\$_{01} + \$_{10}) + (\$_{00} - \$_{11}) \left(\Delta_{\mu 2}^4 - \tfrac{2}{3} \mu_2 p_2 \right) \mu_p^1 \xi \\
&\quad \times \cos 2(\alpha_1 + \alpha_2) \Big] + s_1 s_2 \Big[\chi^{PD} \$_{00} + \eta^{PD} \$_{11} + \Delta_{\mu 2}^2 (\$_{01} + \$_{10}) - (\$_{00} - \$_{11}) \\
&\quad \times \left(\Delta_{\mu 2}^4 - \tfrac{2}{3} \mu_2 p_2 \right) \mu_p^1 \xi \cos 2(\beta_1 + \beta_2) \Big] + s_1 c_2 \Big[\chi^{PD} \$_{01} + \eta^{PD} \$_{10} \\
&\quad + \Delta_{\mu 2}^1 (\$_{00} + \$_{11}) + (\$_{01} - \$_{10}) \left(\Delta_{\mu 2}^4 - \tfrac{2}{3} \mu_2 p_2 \right) \mu_p^1 \xi \cos 2(\alpha_2 - \beta_1) \Big] \\
&\quad + c_1 s_2 \Big[\chi^{PD} \$_{10} + \eta^{PD} \$_{01} + \Delta_{\mu 2}^1 (\$_{00} + \$_{11}) - (\$_{01} - \$_{10}) \left(\Delta_{\mu 2}^4 - \tfrac{2}{3} \mu_2 p_2 \right) \\
&\quad \times \mu_p^1 \xi \cos 2(\alpha_1 - \beta_2) \Big] - \left(\tfrac{1}{4} \mu_p^1 \Delta_{\mu 2}^1 - \tfrac{1}{2} \mu_p^1 \Delta_{\mu 2}^2 + \tfrac{1}{4} \mu_p^1 \Delta_{\mu 2}^3 \right) (\$_{00} + \$_{11}) \\
&\quad \times \sin(\gamma) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) + \left(\tfrac{1}{4} \Delta_{\mu 2}^4 - \tfrac{1}{6} \mu_p^1 \mu_2 p_2 \right) \\
&\quad \times (\$_{00} - \$_{11}) \sin(\delta) \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \\
&\quad + \left(\tfrac{1}{4} \mu_p^1 \Delta_{\mu 2}^1 - \tfrac{1}{2} \mu_p^1 \Delta_{\mu 2}^2 + \tfrac{1}{4} \mu_p^1 \Delta_{\mu 2}^3 \right) (\$_{01} + \$_{10}) \sin(\gamma) \sin(\theta_1) \\
&\quad \times \sin(\theta_2) \sin(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) + \left(\tfrac{1}{4} \Delta_{\mu 2}^4 - \tfrac{1}{6} \mu_p^1 \mu_2 p_2 \right) (\$_{01} - \$_{10}) \sin(\delta) \\
&\quad \times \sin(\theta_1) \sin(\theta_2) \sin(\alpha_1 - \alpha_2 + \beta_1 - \beta_2) \Big].
\end{aligned}
\tag{30}
$$

The definitions of the parameters used in the payoffs, equations (22)–(30), are given in appendix B.
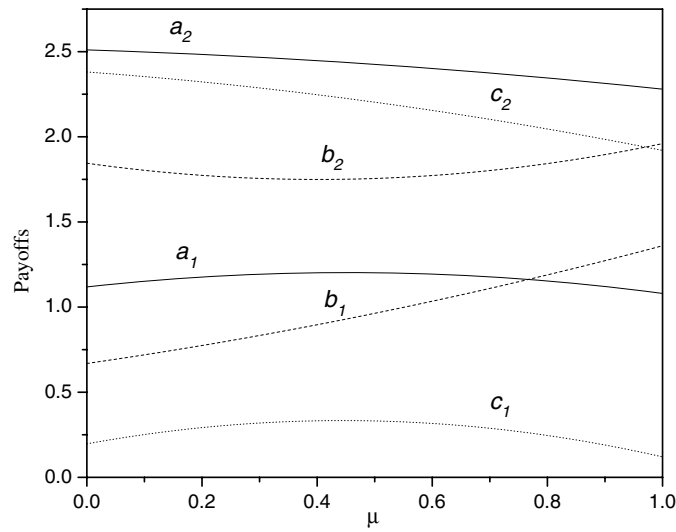
Figure 2. Players (Alice/Bob) payoffs as a function of the memory parameter μ are plotted for the quantum games Prisoner's Dilemma (solid lines, a_i), Battle of the Sexes (dashed lines, b_i) and Chicken (dotted lines, c_i) for amplitude-damping channel. Indices 1 and 2 correspond to the values of the decoherence parameter $p = 0.8$ and $p = 0.2$, respectively, with $\delta = \gamma = 0$, $\theta_1 = 0$, $\theta_2 = \pi/2$ and $\alpha_2 = \pi/2$, $\beta_2 = 0$ as players' optimal strategies.

The payoffs for the two players can be found by substituting the appropriate values for $\$_{ij}$ (elements of the payoff matrix for the corresponding game as given in appendix A) in the above equations. These payoffs become the classical payoffs for $\gamma = \delta = 0$, $p_1 = p_2 = 0$ and $\mu_1 = \mu_2 = 0$. It can be easily proved that for $\gamma = \delta = \pi/2$, with $\beta_1 = \beta_2 = 0$, $\mu_1 = \mu_2 = 0$ and $p_1 = 1$ or $p_2 = 1$, the results of [17] are reproduced for all the nine cases in Prisoner's Dilemma game. Nawaz and Toor have shown that in the case of phase-damping channel, for maximum correlation the effects of decoherence diminish and it behaves as a noiseless game [22]. However, in the case of amplitude and depolarizing channels, for maximum correlation the effects of decoherence persist and cause a reduction in the payoffs and it does not behave as a noiseless game.

3. Results and discussions

To analyze the effects of memory in quantum games, we consider a situation in which Alice is restricted to play classical strategies, i.e., $\alpha_1 = \beta_1 = 0$, whereas Bob is allowed to play the quantum strategies as well. Under these circumstances, the following four cases for the different combinations of δ and γ are worth noting.

Case (i). When $\delta = \gamma = 0$, the payoffs reduce to classical results for unital case, i.e. phase-damping and depolarizing channels. These payoffs, as expected, are independent of the quantum strategies α_2, β_2, but only depend upon the decoherence parameter p and the memory parameter μ. For non-unital case, i.e. amplitude-damping channel, the results reduce to the classical game when we put $p_1 = p_2 = 0$ along with $\mu_1 = \mu_2 = 0$. However, the payoffs of the two players remain independent of quantum phases and the decrease due to decoherence is compensated by the memory and payoffs are enhanced from their classical counterparts (which can be seen from figure 2 for all the three games).
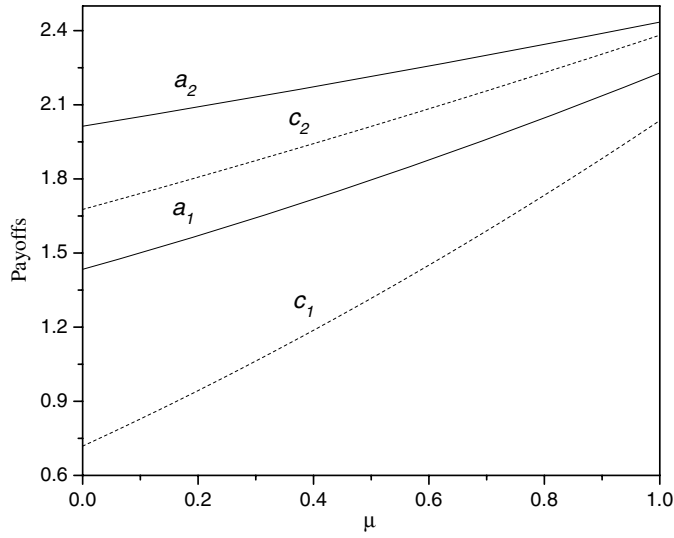
Figure 3. Bob's payoff as a function of the memory parameter μ is plotted for Prisoner's Dilemma (solid lines, a_i) and Chicken (dashed lines, c_i) for amplitude-damping channel. Indices 1 and 2 correspond to the decoherence parameter $p = 0.8$ and $p = 0.2$, respectively, with $\delta = 0, \gamma = \pi/2, \theta_1 = \theta_2 = \pi/2$ and $\alpha_2 = \pi/2, \beta_2 = 0$ as Bob's optimal strategy.

Case (ii). When $\delta = 0, \gamma \neq 0$, and channels 1 and 2 are amplitude-damping.

(a) In the case of Prisoner's Dilemma and Chicken games, the effect of memory can be summarized as: when the decoherence parameter p increases the payoffs start decreasing, however, this effect is partially overcome by the addition of memory, i.e. as μ increases the payoff increases and as a result it compensates the reduction in player's payoffs due to decoherence (as shown in figure 3).

(b) In the case of the Battle of the Sexes game, the quantum player enjoys an advantage over the classical player for $0 < p \leqslant 1$ (as can be seen from figure 4). The optimal strategy for Bob is to play $\alpha_2 = \pi/2$ and $\beta_2 = 0$.

(c) When channels 1 and 2 are phase-damping and amplitude-damping or depolarizing and amplitude-damping, respectively, the quantum player remains superior over the classical player in the Battle of the Sexes game only (which can be seen from figure 4).

(d) When channels 1 and 2 are phase-damping or depolarizing, the payoffs of the players remain equal in all the three games, however, memory controls the payoffs reduction due to decoherence.

Case (iii). When $\gamma = 0, \delta \neq 0$, and channels 1 and 2 are depolarizing.

(a) In the case of the Battle of the Sexes game, the quantum player outperforms the classical player for $0 < p \leqslant 1$ (as can be seen from figure 5). The optimal strategy for Bob is to play $\alpha_2 = 0$ and $\beta_2 = \pi/2$.

(b) For phase-damping or amplitude-damping channels, the payoffs of the players remain equal in all the three games considered and memory compensates the decoherence effects in Prisoner's Dilemma and Chicken games.

(c) It can be seen from figure 5 that when channels 1 and 2 are phase-damping and amplitude-damping or depolarizing and amplitude-damping, respectively, the quantum player remains superior over the classical player for the case of the Battle of the Sexes game.
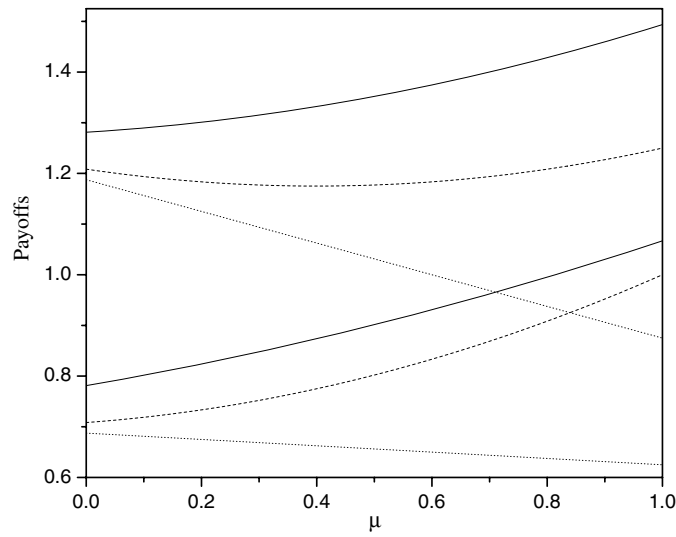
10

Figure 4. Payoffs for Alice (classical player) and Bob (quantum player) are plotted as a function of the memory parameter μ for amplitude-damping (solid lines), depolarizing followed by an amplitude-damping (dashed lines) and phase-damping followed by an amplitude-damping (dotted lines) channels for the Battle of the Sexes game with $\delta = 0$, $\gamma = \pi/2$, $p = 0.5$, $\theta_1 = 0$, $\theta_2 = \pi/2$ and $\alpha_2 = \pi/2$, $\beta_2 = 0$ as Bob's optimal strategy. The lower curves for all the three cases correspond to Alice's payoff.
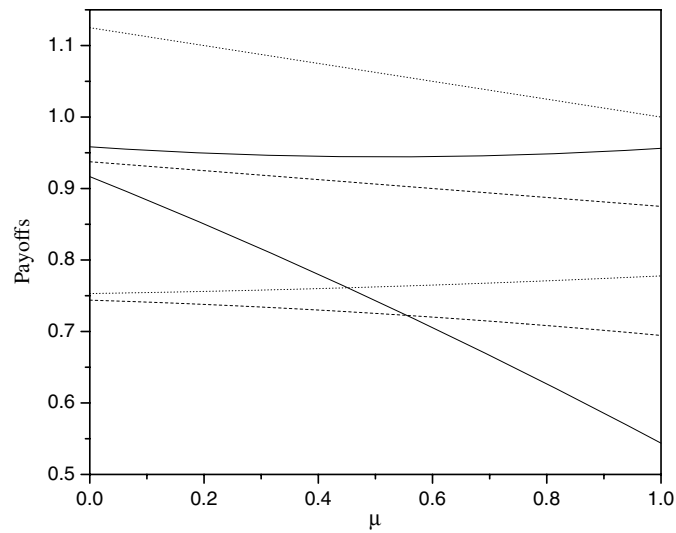

Figure 5. Payoffs for Alice and Bob are plotted as a function of the memory parameter μ for amplitude-damping (solid lines), depolarizing followed by an amplitude-damping (dashed lines) and phase-damping followed by an amplitude-damping (dotted lines) channels for the Battle of the Sexes game with $\gamma = 0$, $\delta = \pi/2$, $p = 0.5$, $\theta_1 = 0$, $\theta_2 = \pi/2$ and $\alpha_2 = 0$, $\beta_2 = \pi/2$ as Bob's optimal strategy. The lower curves for all the three cases correspond to Alice's payoff.

Case (iv). When $\gamma = \delta = \pi/2$, with $\mu = 0$ (memoryless case), the quantum player outscores the classical player for $p < 1$ in all the nine cases. For the memory parameter $\mu \neq 0$, the quantum player outperforms the classical player even for the maximum noise, i.e., $p = 1$,
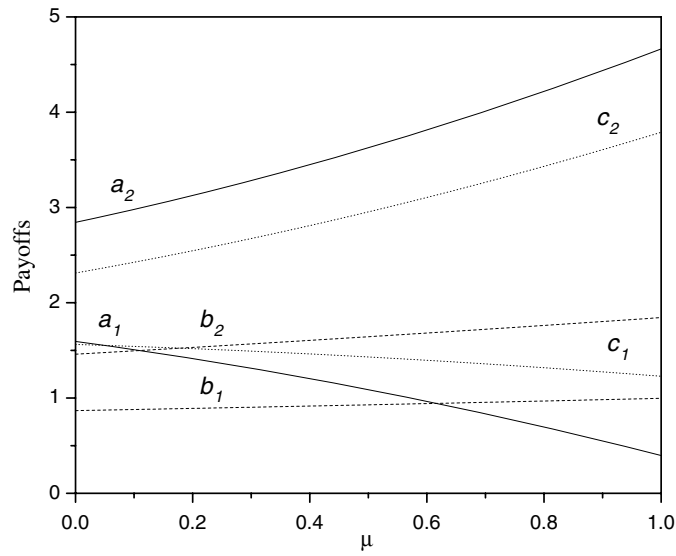
Figure 6. Alice's (1) and Bob's (2) payoffs are plotted as a function of the memory parameter μ for the quantum games Prisoner's Dilemma (solid lines, a_i), Battle of the Sexes (dashed lines, b_i) and Chicken (dotted lines, c_i) for amplitude-damping channel with $\delta = \gamma = \pi/2$, $p = 0.5$, $\theta_1 = 0$, $\theta_2 = \pi/2$ and $\alpha_2 = \pi/2$, $\beta_2 = 0$ as Bob's optimal strategy.

Figure 7. Alice's (1) and Bob's (2) payoffs are plotted as a function of the memory parameter μ for the quantum games Prisoner's Dilemma (solid lines, a_i), Battle of the Sexes (dashed lines, b_i) and Chicken (dotted lines, c_i) for depolarizing channel with $\delta = \gamma = \pi/2$, $p = 0.5$, $\theta_1 = 0$, $\theta_2 = \pi/2$ and $\alpha_2 = \pi/2$, $\beta_2 = 0$ as Bob's optimal strategy.
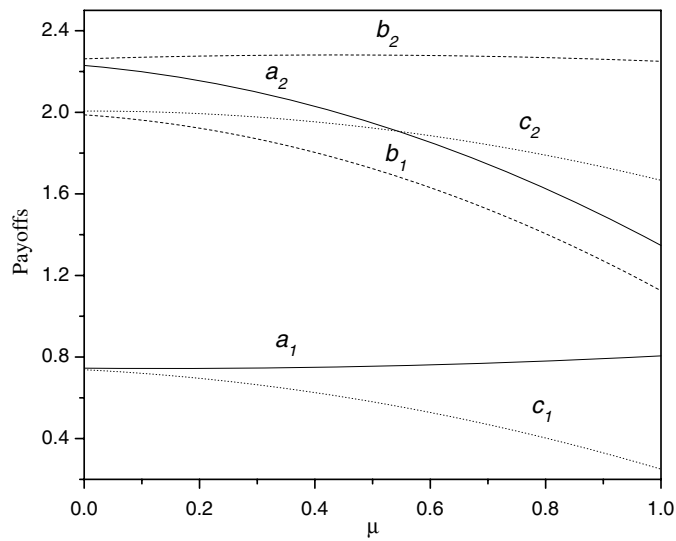
for all the nine cases, which is not possible in the memoryless case. It can be seen from figures 6 and 7 for amplitude-damping and depolarizing channels, respectively. Similar behavior is seen for all the remaining seven channels.

A Nash equilibrium implies that no player can increase his payoff by unilaterally changing his strategy. One can see from case (ii)-b that for Alice $\theta_1 = 0$ and for Bob $\theta_2 = \pi/2$ and $\alpha_2 = \pi/2$, $\beta_2 = 0$ remain their best strategies throughout the course of the game for the entire range of the decoherence parameter p and the memory parameter μ. Similarly, for case (iii)-a, it can be seen that for Alice $\theta_1 = 0$ and for Bob $\theta_2 = \pi/2$ and $\alpha_2 = 0$, $\beta_2 = \pi/2$ remain their best strategies for all values of p and μ and no player can increase his/her payoff by unilaterally changing his/her strategy. A similar situation occurs for all the remaining cases. Thus by inspection (from equations (22) to (30)), one can see that the Nash equilibria of the three games do not change under the effect of memory.

4. Conclusions

Quantum games with correlated noise are studied under the generalized quantization scheme [24]. Three games, Prisoner's Dilemma, Battle of the Sexes and Chicken, are studied with one player restricted to the classical strategies, while the other is allowed to play quantum strategies. It is shown that the effects of the memory and decoherence become effective for the case $\gamma = \delta = \pi/2$, for which the quantum player outperforms the classical player in all the three games for maximally entangled case. It is also shown that the quantum player enjoys an advantage over the classical player for $\delta = 0$, $\gamma \neq 0$ and $\gamma = 0$, $\delta \neq 0$ cases in the Battle of the Sexes game when amplitude-damping and depolarizing channels are used, respectively. It can be seen that the Nash equilibria of the three games do not change under the effect of memory.

Appendix A. Classical games

A brief description of the three classical games, Prisoner's Dilemma, Battle of the Sexes and Chicken, is given below.

A.1. Prisoner's Dilemma

This game depicts a situation where two prisoners, who have committed a crime together, are being interrogated in separate cells. The two possible moves for each prisoner are, to cooperate (C) or to defect (D). They are not allowed to communicate but have access to the following payoff matrix:

$$
\begin{array}{c}
 & \text{Bob} \\
 & \begin{array}{cc} C & D \end{array} \\
\text{Alice} \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{bmatrix}.
\end{array}
\tag{A.1}
$$

It is clear from the payoff matrix (A.1) that D is the dominant strategy for the two players. Therefore, rational reasoning forces the players to play D. Hence (D, D) is the Nash equilibrium of the game with payoffs (1,1). But the players could get higher payoffs if they would have played C instead of D. This is the dilemma of the game.

A.2. Battle of the Sexes

The payoff matrix for the Battle of the Sexes game is

$$
\begin{array}{c}
 & \text{Bob} \\
 & \begin{array}{cc} O & T \end{array} \\
\text{Alice} \begin{array}{c} O \\ T \end{array} & \begin{bmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{bmatrix}.
\end{array}
\tag{A.2}
$$

In this game Alice is fond of Opera whereas Bob likes watching TV, but they also want to spend the evening together. The two pure Nash equilibria (NE) of this game are (O, O) and (T, T) which correspond to the situation when both the players choose Opera and TV, respectively. Here the first NE is favorable to Alice, while the second NE is favorable to Bob. Since they are not allowed to communicate, they face a dilemma in choosing their strategies.

A.3. Chicken game

The payoff matrix for the Chicken game is

$$
\begin{array}{c}
& \text{Bob} \\
& \begin{array}{cc} C & \quad D \end{array} \\
\text{Alice}\begin{array}{c} C \\ D \end{array} & \left[\begin{array}{cc} (3,3) & (1,4) \\ (4,1) & (0,0) \end{array}\right].
\end{array} \tag{A.3}
$$

In the game of Chicken, also known as the Hawk–Dove game, two players drove their cars towards each other. The first one who swerves to avoid collision is the loser (chicken) and the second one who keeps on driving straight is the winner. There is no dominant strategy in this game. There are two NE, (C, D) and (D, C), the former is preferred by Bob and the latter by Alice. The dilemma of this game is that the Pareto Optimal strategy (C, C) is not the Nash equilibrium.

Appendix B. Some definitions

The definitions of the parameters used in equation (22) are given as

$$\eta_1^A = \chi_{\mu_1}^{(00)} \chi_{\mu_2}^{(00)} \cos^2(\gamma/2) \cos^2(\delta/2) + (\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)} \cos^2(\gamma/2)) \sin^2(\delta/2)$$
$$+ \left(\chi_{\mu_1}^{(00)} \chi_{\mu_2}^{(11)} + 2\chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(a)}\right) \sin^2(\delta/2) \cos^2(\gamma/2),$$

$$\eta_2^A = \chi_{\mu_2}^{(00)} \left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)} \cos^2(\gamma/2)\right) \cos^2(\delta/2) + \left(\chi_{\mu_1}^{(00)} + 2\chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(a)}\right)$$
$$\times \sin^2(\delta/2) \cos^2(\gamma/2) + \chi_{\mu_2}^{(11)} \left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)} \cos^2(\gamma/2)\right) \sin^2(\delta/2),$$

$$\eta_3^A = \chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(00)} \cos^2(\gamma/2) \cos^2(\delta/2) + \left(\chi_{\mu_1}^{(01)} + \chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(11)}\right) \cos^2(\gamma/2) \sin^2(\delta/2)$$
$$+ \chi_{\mu_2}^{(a)} \left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)} \cos^2(\gamma/2) + \chi_{\mu_1}^{(00)} \cos^2(\gamma/2)\right) \sin^2(\delta/2),$$

$$\eta_4^A = \chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(00)} \cos^2(\delta/2) + \left(\chi_{\mu_1}^{(10)} + \chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(11)} - 2\chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(a)}\right) \sin^2(\delta/2),$$

$$\chi_1^A = \chi_{\mu_1}^{(00)} \chi_{\mu_2}^{(00)} \cos^2(\gamma/2) \sin^2(\delta/2) + \left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)} \cos^2(\gamma/2)\right) \cos^2(\delta/2)$$
$$+ \left(\chi_{\mu_1}^{(00)} \chi_{\mu_2}^{(11)} + 2\chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(a)}\right) \cos^2(\delta/2) \cos^2(\gamma/2),$$

$$\chi_2^A = \chi_{\mu_2}^{(00)} \left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)} \cos^2(\gamma/2)\right) \sin^2(\delta/2) + \left(\chi_{\mu_1}^{(00)} + 2\chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(a)}\right)$$
$$\times \cos^2(\delta/2) \cos^2(\gamma/2) + \chi_{\mu_2}^{(11)} \left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)} \cos^2(\gamma/2)\right) \cos^2(\delta/2),$$

$$\chi_3^A = \left(\chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(00)} + \chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(11)}\right) \cos^2(\gamma/2) \sin^2(\delta/2) + \chi_{\mu_1}^{(01)} \cos^2(\gamma/2) \cos^2(\delta/2)$$
$$+ \chi_{\mu_2}^{(a)} \left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)} \cos^2(\gamma/2) + \chi_{\mu_1}^{(00)} \cos^2(\gamma/2)\right) \cos^2(\delta/2),$$

$$\chi_4^A = \chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(00)} \sin^2(\delta/2) + \left(\chi_{\mu_1}^{(10)} + \chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(11)} - 2\chi_{\mu_1}^{(10)} \chi_{\mu_2}^{(a)}\right) \cos^2(\delta/2),$$

$$\Delta_1^A = \left(\chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(b)} + \chi_{\mu_1}^{(00)} \chi_{\mu_2}^{(01)}\right) \cos^2(\gamma/2),$$

$$\Delta_2^A = \chi_{\mu_1}^{(01)} \chi_{\mu_2}^{(b)} \cos^2(\gamma/2) + \chi_{\mu_2}^{(01)} \left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)} \cos^2(\gamma/2)\right),$$

$$\Delta_3^A = \chi_{\mu_2}^{(b)}\left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)}\cos^2(\gamma/2)\right)\cos^2(\delta/2)$$
$$+ \left(\chi_{\mu_1}^{(01)}\chi_{\mu_2}^{(01)} + \chi_{\mu_1}^{(00)}\chi_{\mu_2}^{(b)}\sin^2(\delta/2)\right)\cos^2(\gamma/2),$$

$$\Delta_4^A = \chi_{\mu_2}^{(b)}\left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)}\cos^2(\gamma/2)\right)\sin^2(\delta/2)$$
$$+ \left(\chi_{\mu_1}^{(01)}\chi_{\mu_2}^{(01)} + \chi_{\mu_1}^{(00)}\chi_{\mu_2}^{(b)}\cos^2(\delta/2)\right)\cos^2(\gamma/2),$$

$$\Delta_5^A = \left(\sin^2(\gamma/2) + \chi_{\mu_1}^{(11)}\cos^2(\gamma/2)\right) + \chi_{\mu_1}^{(00)} - 2\chi_{\mu_1}^{(01)}\cos^2(\gamma/2),$$

$$\Delta_6^A = \chi_{\mu_1}^{(10)}\chi_{\mu_2}^{(01)} - \chi_{\mu_1}^{(10)}\chi_{\mu_2}^{(b)}, \qquad \xi = \frac{1}{2}\sin(\delta)\sin(\gamma),$$

$$s_i = \sin^2\left(\frac{\theta_i}{2}\right), \qquad c_i = \cos^2\left(\frac{\theta_i}{2}\right),$$

$$\chi_{\mu_1}^{(00)} = (1-p_1)^2 + \mu_1(1-p_1)p_1,$$

$$\chi_{\mu2}^{(00)} = (1-p_2)^2 + \mu_2(1-p_2)p_2,$$

$$\chi_{\mu_1}^{(11)} = p_1^2 + \mu_1(1-p_1)p_1,$$

$$\chi_{\mu_2}^{(11)} = p_2^2 + \mu_2(1-p_2)p_2,$$

$$\chi_{\mu_1}^{(10)} = (1-\mu_1)(1-p_1) + \mu_1(1-p_1)^{\frac{1}{2}},$$

$$\chi_{\mu2}^{(10)} = (1-\mu_2)(1-p_2) + \mu_2(1-p_2)^{\frac{1}{2}},$$

$$\chi_{\mu_1}^{(01)} = (1-\mu_1)(1-p_1)p_1, \qquad \chi_{\mu2}^{(01)} = (1-\mu_2)(1-p_2)p_2,$$

$$\chi_{\mu_2}^{(a)} = (1-\mu_2)p_2, \qquad \chi_{\mu_2}^{(b)} = (1-p_2) + \mu_2 p_2.$$

The definitions of the parameters used in equation (23) are given as

$$\Delta_{\mu1}^1 = -\frac{1}{9}(-3+2p_1)(-2p_1+2\mu_1 p_1 + 3),$$

$$\Delta_{\mu1}^2 = -\frac{2}{9}p_1(-2p_1+2\mu_1 p_1 - 3\mu_1),$$

$$\Delta_{\mu1}^3 = -\frac{1}{9}\left(-9+24p_1-18\mu_1 p_1 - 16p_1^2 + 16\mu_1 p_1^2\right) - \frac{2}{3}\mu_1 p_1,$$

$$\Delta_{\mu1}^4 = \frac{2}{9}p_1(-3+2p_1)(\mu_1 - 1),$$

$$\Delta_{\mu2}^1 = -\frac{1}{9}(-3+2p_2)(-2p_2+2\mu_2 p_2 + 3),$$

$$\Delta_{\mu2}^2 = \frac{2}{9}p_2(-3+2p_2)(\mu_2 - 1),$$

$$\Delta_{\mu2}^3 = -\frac{2}{9}p_2(-2p_2+2\mu_2 p_2 - 3\mu_2),$$

$$\Delta_{\mu2}^4 = -\frac{1}{9}\left(-9+24p_2-18\mu_2 p_2 - 16p_2^2 + 16\mu_2 p_2^2\right),$$

$$\Delta_\mu^{11} = \Delta_{\mu1}^1\cos^2(\gamma/2) + \Delta_{\mu1}^2\sin^2(\gamma/2),$$

$$\Delta_\mu^{21} = \Delta_{\mu1}^2\cos^2(\gamma/2) + \Delta_{\mu1}^1\sin^2(\gamma/2),$$

$$\eta^D = \left(\Delta_{\mu2}^1\Delta_\mu^{11} + \Delta_{\mu2}^3\Delta_\mu^{21}\right)\cos^2(\delta/2) + \left(\Delta_{\mu2}^1\Delta_\mu^{21} + \Delta_{\mu2}^3\Delta_\mu^{11}\right)\sin^2(\delta/2) + 2\Delta_{\mu2}^2\Delta_{\mu1}^4,$$

$$\chi^D = \left(\Delta_{\mu2}^1\Delta_\mu^{21} + \Delta_{\mu2}^3\Delta_\mu^{11}\right)\cos^2(\delta/2) + \left(\Delta_{\mu2}^1\Delta_\mu^{11} + \Delta_{\mu2}^3\Delta_\mu^{21}\right)\sin^2(\delta/2) + 2\Delta_{\mu2}^2\Delta_{\mu1}^4,$$

$$\Delta^D = \Delta_{\mu2}^2\Delta_\mu^{11} + \Delta_{\mu2}^2\Delta_\mu^{21} + \Delta_{\mu2}^1\Delta_{\mu1}^4 + \Delta_{\mu2}^3\Delta_{\mu1}^4,$$

$$\eta_{1DP} = -\left(\Delta_{\mu1}^2\cos^2(\gamma/2) + \Delta_{\mu1}^1\sin^2(\gamma/2)\right) - \left(\Delta_{\mu1}^2\sin^2(\gamma/2) + \Delta_{\mu1}^1\cos^2(\gamma/2)\right) + 2\Delta_{\mu1}^4.$$

The definitions of the parameters used in equation (24) are given as

$$\eta^P = \cos^2(\gamma/2)\cos^2(\delta/2) + \sin^2(\gamma/2)\sin^2(\delta/2),$$

$$\chi^P = \sin^2(\gamma/2)\cos^2(\delta/2) + \cos^2(\gamma/2)\sin^2(\delta/2),$$

$$\mu_p^{(i)} = (1-\mu_i)(1-p_i)^2 + \mu_i.$$

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The definitions of the parameters used in equation (25) are given as

$$\eta_1^{PA} = \chi_{\mu2}^{(00)} \cos^2(\gamma/2) \cos^2(\delta/2) + \left(\sin^2(\gamma/2) + \chi_{\mu2}^{(11)} \cos^2(\gamma/2)\right) \sin^2(\delta/2),$$

$$\eta_2^{PA} = \left(\cos^2(\gamma/2) + \chi_{\mu2}^{(11)} \sin^2(\gamma/2)\right) \sin^2(\delta/2) + \chi_{\mu2}^{(00)} \sin^2(\gamma/2) \cos^2(\delta/2),$$

$$\eta_3^{PA} = \chi_{\mu2}^{(a)} \sin^2(\gamma/2) \sin^2(\delta/2) + \chi_{\mu2}^{(a)} \cos^2(\gamma/2) \sin^2(\delta/2),$$

$$\chi_1^{PA} = (\cos^2(\gamma/2) + \chi_{\mu2}^{(11)} \sin^2(\gamma/2)) \cos^2(\delta/2) + \chi_{\mu2}^{(00)} \cos^2(\gamma/2) \sin^2(\delta/2),$$

$$\chi_2^{PA} = \chi_{\mu2}^{(00)} \sin^2(\gamma/2) \sin^2(\delta/2) + \left(\cos^2(\gamma/2) + \chi_{\mu2}^{(11)} \sin^2(\gamma/2)\right) \cos^2(\delta/2),$$

$$\chi_3^{PA} = \chi_{\mu2}^{(a)} \sin^2(\gamma/2) \cos^2(\delta/2) + \chi_{\mu2}^{(a)} \cos^2(\gamma/2) \cos^2(\delta/2),$$

$$\Delta_1^{PA} = \chi_{\mu2}^{(01)} \cos^2(\gamma/2) \cos^2(\delta/2) + \chi_{\mu2}^{(01)} \cos^2(\gamma/2) \sin^2(\delta/2),$$

$$\Delta_2^{PA} = \chi_{\mu2}^{(01)} \sin^2(\gamma/2) \cos^2(\delta/2) + \chi_{\mu2}^{(01)} \sin^2(\gamma/2) \sin^2(\delta/2),$$

$$\Delta_3^{PA} = \chi_{\mu2}^{(b)} \sin^2(\gamma/2) \cos^2(\delta/2) + \chi_{\mu2}^{(b)} \cos^2(\gamma/2) \sin^2(\delta/2),$$

$$\Delta_4^{PA} = \chi_{\mu2}^{(b)} \cos^2(\gamma/2) \cos^2(\delta/2) + \chi_{\mu2}^{(b)} \sin^2(\gamma/2) \sin^2(\delta/2).$$

The definitions of the parameters used in equation (26) are given as

$$\eta^{AP} = \chi_{\mu1}^{(00)} \cos^2(\gamma/2) \cos^2(\delta/2) + \left(\sin^2(\gamma/2) + \chi_{\mu1}^{(11)} \cos^2(\gamma/2)\right) \sin^2(\delta/2),$$

$$\chi^{AP} = \left(\sin^2(\gamma/2) + \chi_{\mu1}^{(11)} \cos^2(\gamma/2)\right) \cos^2(\delta/2) + \chi_{\mu1}^{(00)} \cos^2(\gamma/2) \sin^2(\delta/2),$$

$$\Delta^{AP} = \chi_{\mu1}^{(01)} \cos^2(\gamma/2) \cos^2(\delta/2) + \chi_{\mu1}^{(01)} \cos^2(\gamma/2) \sin^2(\delta/2).$$

The definitions of the parameters used in equation (27) are given as

$$\eta^{AD} = \left(\chi_{\mu1}^{(00)} \Delta_{\mu2}^1 \cos^2(\gamma/2) + \Delta_{\mu2}^3\left(\sin^2(\gamma/2) + \chi_{\mu1}^{(11)} \cos^2(\gamma/2)\right)\right) \cos^2(\delta/2) + \left(\Delta_{\mu2}^1(\sin^2(\gamma/2)\right.$$
$$\left. + \chi_{\mu1}^{(11)} \cos^2(\gamma/2)) + \chi_{\mu1}^{(00)} \Delta_{\mu2}^3 \cos^2(\gamma/2)\right) \sin^2(\delta/2) + 2\chi_{\mu1}^{(01)} \Delta_{\mu2}^2 \cos^2(\gamma/2),$$

$$\chi^{AD} = \left(\chi_{\mu1}^{(00)} \Delta_{\mu2}^1 \cos^2(\gamma/2) + \Delta_{\mu2}^3\left(\sin^2(\gamma/2) + \chi_{\mu1}^{(11)} \cos^2(\gamma/2)\right)\right) \sin^2(\delta/2) + \left(\Delta_{\mu2}^1(\sin^2(\gamma/2)\right.$$
$$\left. + \chi_{\mu1}^{(11)} \cos^2(\gamma/2)) + \chi_{\mu1}^{(00)} \Delta_{\mu2}^3 \cos^2(\gamma/2)\right) \cos^2(\delta/2) + 2\chi_{\mu1}^{(01)} \Delta_{\mu2}^2 \cos^2(\gamma/2),$$

$$\Delta^{AD} = \chi_{\mu1}^{(01)}\left(\Delta_{\mu2}^1 + \Delta_{\mu2}^3\right) \cos^2(\gamma/2) + \Delta_{\mu2}^2\left(\sin^2(\gamma/2) + \chi_{\mu1}^{(11)} \cos^2(\gamma/2)\right) + \chi_{\mu1}^{(00)} \Delta_{\mu2}^2 \cos^2(\gamma/2),$$

$$\eta_{1AD} = \chi_{\mu1}^{(11)} \cos^2(\gamma/2)) + \chi_{\mu1}^{(00)} \cos^2(\gamma/2) - 2\chi_{\mu1}^{(01)} \cos^2(\gamma/2) + \sin^2(\gamma/2).$$

The definitions of the parameters used in equation (28) are given as

$$\eta_1^{DA} = \Delta_\mu^{11} \chi_{\mu2}^{(00)} \cos^2(\delta/2) + \Delta_\mu^{11} \chi_{\mu2}^{(11)} \sin^2(\delta/2) + \Delta_\mu^{12} \sin^2(\delta/2) + 2\Delta_{\mu1}^4 \chi_{\mu2}^{(a)} \sin^2(\delta/2),$$

$$\eta_2^{DA} = \Delta_\mu^{21} \chi_{\mu2}^{(00)} \cos^2(\delta/2) + \Delta_\mu^{21} \chi_{\mu2}^{(11)} \sin^2(\delta/2) + \Delta_\mu^{11} \sin^2(\delta/2) + 2\Delta_{\mu1}^4 \chi_{\mu2}^{(a)} \sin^2(\delta/2),$$

$$\eta_3^{DA} = \left(\Delta_\mu^{11} \chi_{\mu2}^{(a)} + \Delta_\mu^{21} \chi_{\mu2}^{(a)}\right) \sin^2(\delta/2) + \Delta_{\mu1}^4 \chi_{\mu2}^{(00)} \cos^2(\delta/2) + \left(\chi_{\mu2}^{(11)} + 1\right) \Delta_{\mu1}^4 \sin^2(\delta/2),$$

$$\chi_1^{DA} = \Delta_\mu^{11} \chi_{\mu2}^{(00)} \sin^2(\delta/2) + \Delta_\mu^{11} \chi_{\mu2}^{(11)} \cos^2(\delta/2) + \Delta_\mu^{12} \cos^2(\delta/2) + 2\Delta_{\mu1}^4 \chi_{\mu2}^{(a)} \cos^2(\delta/2),$$

$$\chi_2^{DA} = \Delta_\mu^{21} \chi_{\mu2}^{(00)} \sin^2(\delta/2) + \Delta_\mu^{21} \chi_{\mu2}^{(11)} \cos^2(\delta/2) + \Delta_\mu^{11} \cos^2(\delta/2) + 2\Delta_{\mu1}^4 \chi_{\mu2}^{(a)} \cos^2(\delta/2),$$

$$\chi_3^{DA} = \left(\Delta_\mu^{11} \chi_{\mu2}^{(a)} + \Delta_\mu^{21} \chi_{\mu2}^{(a)}\right) \cos^2(\delta/2) + \Delta_{\mu1}^4 \chi_{\mu2}^{(00)} \sin^2(\delta/2) + \left(\chi_{\mu2}^{(11)} + 1\right) \Delta_{\mu1}^4 \cos^2(\delta/2),$$

$$\Delta_1^{DA} = \Delta_\mu^{11} \chi_{\mu2}^{(01)} + \Delta_{\mu1}^4 \chi_{\mu2}^{(b)},$$

$$\Delta_2^{DA} = \Delta_\mu^{11} \chi_{\mu2}^{(b)} \sin^2(\delta/2) + \Delta_\mu^{21} \chi_{\mu2}^{(b)} \cos^2(\delta/2) + \Delta_{\mu1}^4 \chi_{\mu2}^{(01)},$$

$$\Delta_3^{DA} = \Delta_\mu^{11} \chi_{\mu2}^{(b)} \cos^2(\delta/2) + \Delta_\mu^{21} \chi_{\mu2}^{(b)} \sin^2(\delta/2) + \Delta_{\mu1}^4 \chi_{\mu2}^{(01)}.$$

The definitions of the parameters used in equation (29) are given as

$$\eta^{DP} = \Delta_\mu^{11} \cos^2(\delta/2) + \Delta_\mu^{12} \sin^2(\delta/2),$$

$$\chi^{DP} = \Delta_\mu^{11} \sin^2(\delta/2) + \Delta_\mu^{12} \cos^2(\delta/2).$$

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The definitions of the parameters used in equation (30) are given as

$$\eta^{PD} = \left(\Delta_{\mu 2}^1 \cos^2(\gamma/2) + \Delta_{\mu 2}^3 \sin^2(\gamma/2)\right) \cos^2(\delta/2)$$
$$+ \left(\Delta_{\mu 2}^1 \sin^2(\gamma/2) + \Delta_{\mu 2}^3 \cos^2(\gamma/2)\right) \sin^2(\delta/2),$$
$$\chi^{PD} = \left(\Delta_{\mu 2}^1 \sin^2(\gamma/2) + \Delta_{\mu 2}^3 \cos^2(\gamma/2)\right) \cos^2(\delta/2)$$
$$+ \left(\Delta_{\mu 2}^1 \cos^2(\gamma/2) + \Delta_{\mu 2}^3 \sin^2(\gamma/2)\right) \sin^2(\delta/2).$$

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